X WWLET WAVELETS AND APPLICATIONS.
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Multiscale approximations and applications

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Sketch of the lecture

1. Subdivision schemes, decimation schemes and associated multi-resolutions
2. Construction of decimation schemes associated to a given subdivision scheme
3. Construction of the details operators
4. Properties of multi-resolutions and applications to compression
5. Application: Convergence of derivatives
6. About divided differences
7. Key properties

1) Subdivisions, decimations and associated multi-resolutions

Subdivision:$\mathbf{h}: f^0 \mapsto \{f^0, f^1, ..., f^j, ...\},\ f^j = (f^j_k)_{k\in\mathbb{Z}} \in l^\infty((\mathbb{Z})$

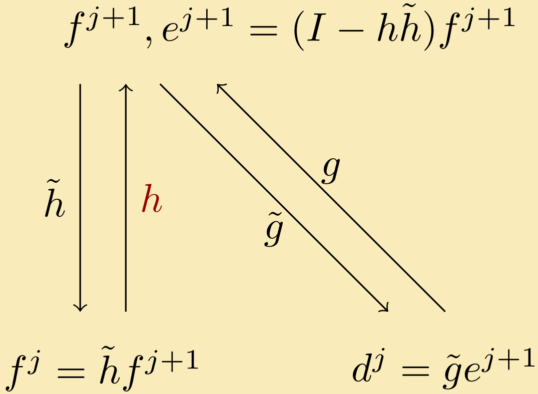
with $h \begin{cases} l^\infty(\mathbb{Z}) & \rightarrow & l^\infty(\mathbb{Z}) \\ f^{j-1} & \mapsto & f^j = h(f^{j-1}) \end{cases}$ h being local and r-shift invariant $(\theta^r hf = h\theta f$ for $(\theta f)_k = f_{k+1}$

Decimation: $\tilde{\mathbf{h}}: f^j \mapsto \{f^{j-1}, f^{j-2}, ..\}$,

with $\tilde{h} \begin{cases} l^\infty(\mathbb{Z}) & \rightarrow & l^\infty(\mathbb{Z}) \\ f^j & \mapsto & f^j = \tilde{h}(f^j) \end{cases}$ $\tilde{h}$ being local and r-shift invariant $(\theta\tilde{h}f = \tilde{h}\theta^r f$ for (from now $r = 2$)

- Linear subdivision: $f^1_k = \sum a_{k-2l} f_l$, References: N. Dyn (1992), A.S Cavaretta et al.(1991)
- Linear decimation: $f^0_k = \sum \tilde{a}_{l-2k} f_l$.

1) Subdivisions, decimations and associated multi-resolutions



$$\mathbf{M} : \{f^0, d^0, d^1, ..., d^j\} \mapsto f^{j+1}.$$

Consistency:
$$\begin{cases} (I - g\tilde{g})(I - h\tilde{h}) & = & 0, \\ I - \tilde{h}h & = & 0, \\ \tilde{h}g & = & 0. \end{cases}$$

1) Subdivisions, decimations and associated multi-resolution

Wellknown situations

Interpolatory subdivision scheme $\Longrightarrow \tilde{h}$ is the subsampling operator$f_k^j = f_{2k}^{j+1}$, details are differences at odd positions

Linear decimation and subdivision schemes are constructed together (Wavelet multiscale analysis): consistency $\Longrightarrow e^{j+1} \in \text{Ker}\tilde{h}$ and g is a projection on Ker$\tilde{h}$ or any isomorph space.

What about other situations?

1) Subdivisions, decimations and associated multi-resolution

Other situations

- Incorporate data information into multiscale transform: data fitted schemes, position dependent schemes, data dependent schemes
- Incorporate nonlinear constraint into the multiscale transform ($\forall j, f^j \in \mathbb{M}$)

2) Construction of decimation schemes associated to a given subdivision scheme: Uniform linear subdivision

Proposition. (Linear scheme) *References: Kui et al. (2016)*

Let h be a linear subdivision of mask $\{h_{n-2\alpha}, h_{n-2\alpha+1}, \ldots, h_n, h_{n+1}\}$

$$H = \begin{bmatrix} h_n & h_{n-2} & \cdots & h_{n-2\alpha} & 0 & \cdots & 0 \\ h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & 0 & \cdots & 0 \\ 0 & h_n & h_{n-2} & \cdots & h_{n-2\alpha} & \cdots & 0 \\ 0 & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} & \cdots & 0 \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & h_n & h_{n-2} & \cdots & h_{n-2\alpha} \\ 0 & 0 & \cdots & h_{n+1} & h_{n-1} & \cdots & h_{n-2\alpha+1} \end{bmatrix}.$$

If $\det(H) \neq 0$, there exists at most 2α consistent elementary decimation operators whose masks are of length not larger than 2α. These masks are given by each row of H^{-1}.

2) Construction of decimation schemes associated to a given subdivision scheme: General linear subdivision

Proposition. (Generation of all consistent decimations)

Let $\{\tilde{h}^i\}_{1\leq i\leq 2\alpha}$ be the set of elementary consistent decimation operators.

For any decimation operator $\tilde{h}$ constructed from $(\tilde{h}_k)_{k\in\mathbb{Z}}$ and any integer t, we define $T_t(\tilde{h})$ the decimation operator related to the sequence $(\tilde{h}_{k-t})_{k\in\mathbb{Z}}$.

Then, all the consistent decimation operators can be constructed as

$$\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{I}} c_{i,t} T_{2t}(\tilde{h}^i),$$

with

$$\forall t \in \mathcal{T} \subset \mathbb{Z}, \sum_{i\in\mathcal{I}} c_{i,t} = \delta_{t,0}, \quad \text{and } 0 \in \mathcal{T} \quad .$$

2) Construction of decimation schemes associated to a given linear subdivision scheme

Example of shifted Lagrange subdivision

Definition (degree 3) References: Dyn et al. (2004)

$$P_k(x) = L_{-1}(x) f_{k-1} + L_0(x) f_k + L_1(x) f_{k+1} + L_2(x) f_{k+2}.$$

where $\{L_n(x)\}_{-1 \leq n \leq 2}$ denotes the degree 3 Lagrange interpolatory function associated to the stencil $\{-1, 0, 1, 2\}$.

$$\begin{cases} (h_L f)_{2k} & = & P_k(\frac{1}{4}) \\ (h_L f)_{2k+1} & = & P_k(\frac{3}{4}). \end{cases}$$

Decimations associated to the Shifted lagrange subdivision

Mask of the Shifted lagrange subdivision

$$M_h = \{h_k, -4 \le k \le 3\} = \{-\frac{5}{128}, -\frac{7}{128}, \frac{35}{128}, \frac{105}{128}, \frac{105}{128}, \frac{35}{128}, -\frac{7}{128}, -\frac{5}{128}\}.$$

Matrix of the correspondant consistent elementary decimations

$$\tilde{H} =$$

$$\begin{pmatrix} \tilde{h}_0 \\ \tilde{h}_2 \\ \tilde{h}_4 \\ \tilde{h}_6 \\ \tilde{h}_8 \\ \tilde{h}_{10} \end{pmatrix} = \begin{bmatrix} \frac{24367}{1152} & -\frac{63605}{1152} & \frac{31115}{576} & -\frac{10325}{576} & -\frac{4165}{1152} & \frac{2975}{1152} \\ \frac{2975}{1152} & -\frac{4165}{1152} & \frac{1771}{576} & -\frac{565}{576} & -\frac{245}{1152} & \frac{175}{1152} \\ \frac{175}{1152} & -\frac{245}{1152} & \frac{875}{576} & -\frac{245}{576} & -\frac{133}{1152} & \frac{95}{1152} \\ \frac{95}{1152} & -\frac{133}{1152} & -\frac{245}{576} & \frac{875}{576} & -\frac{245}{1152} & \frac{175}{1152} \\ \frac{175}{1152} & -\frac{245}{1152} & -\frac{565}{576} & \frac{1771}{576} & -\frac{4165}{1152} & \frac{2975}{1152} \\ \frac{2975}{1152} & -\frac{4165}{1152} & -\frac{10325}{576} & \frac{31115}{576} & -\frac{63605}{1152} & \frac{24367}{1152} \end{bmatrix}.$$

2) Construction of decimation schemes associated to a given subdivision scheme:

Non linear subdivision scheme written as a perturbation of a linear scheme

$f^{j+1} = hf^j = h_L f^j + h_N f^j$

Applying $\tilde{h}_L$ we get a fixed point relation:

$$f^j = \tilde{h}_L f^{j+1} - \tilde{h}_L h_N f^j.$$

Proposition.

If h_L is such that $\tilde{h}_L h_N$ is contractive then the above formula defines a non linear decimation operator consistent with h.

2) Construction of decimation schemes associated to a given non linear subdivision scheme: Example of the Shifted PPH subdivision scheme

Let

$A(x,y) = \frac{x+y}{2}, H(x,y) = \frac{xy}{x+y}\left(sgn(xy)+1\right), D_k = f_{k+1} - 2f_k + f_{k-1},$

Define N_k as:

if $|D_k| \leq |D_{k+1}|$, $N_k(x) = 2L_2(x)\left(H(D_k, D_{k+1}) - A(D_k, D_{k+1})\right),$

if $|D_k| > |D_{k+1}|$, $N_k(x) = 2L_{-1}(x)\left(H(D_k, D_{k+1}) - A(D_k, D_{k+1})\right),$

and

$$\begin{cases} (h_N f)_{2k} & = & N_k(\frac{1}{4}), \\ (h_N f)_{2k+1} & = & N_k(\frac{3}{4}), \end{cases}$$

then $hf = h_L f + h_N f$ defines the shifted PPH subdivision scheme

References: Amat et al. (2011)

Example of the Shifted PPH subdivision scheme

Moreover, for $((\tilde{h}_L)_k, -4 \leq k \leq 3) = \frac{1}{2}\tilde{h}_4 + \frac{1}{2}\tilde{h}_6 =$ $(\frac{95}{2304}, -\frac{133}{2304}, -\frac{35}{256}, \frac{1505}{2304}, \frac{1505}{2304}, -\frac{35}{256}, -\frac{133}{2304}, \frac{95}{2304})$, the operator $\tilde{h}_L h_N$ is contractive and a consistent decimation $\tilde{h}$ is therefore available.

3) Construction of the details operators

<u>Definition of the details</u>

Since $e^{j+1} = (I - h\tilde{h})f^{j+1}$,we have $\tilde{h}_L e^{j+1} = 0$. Therefore there exists square matrices M, N formed using $M_{\tilde{h}_L}$ such that

$$Me^{j+1}_{even} = Ne^{j+1}_{odd}.$$

If N invertible one can define $d^j_k = e^{j+1}_{2k}$, and we have

$$e^{j+1}_{even} = d^j, \quad e^{j+1}_{odd} = N^{-1}Md^j.$$

4) Properties of multiresolutions and numerical applications

- Convergence and stability of subdivisions (h_L and h), References: Dyn et al. (2004) , Amat et al. (2011)
- Decay of the errors References: Daubechies (1992) ,
- Stability of the decimations (($\tilde{h}_L, \tilde{h}$),
- Performance for image compression.

4) Properties of multiresolutions and numerical applications

Decay of the errors

Proposition. *(Linear multiresolution) Let* h *be a linear uniform stable subdivision operator and* $\tilde{h}$ *be a linear stable and consistent decimation operator. If* h *quasi reproduces polynomials up to degree* p*, there exist a constant* C *such that for all* $j \in \mathbb{Z}$, $||e^j|| \leq C 2^{-(p+1)j}$.

Decay of the prediction error

Proposition. *(Non linear multiresolution)*

Let h be a non-linear subdivision scheme with $h = h^L + h^N$ where h^L is a linear subdivision quasi-reproducing polynomial of degree p. If, for all $f^j \in l^\infty(\mathbb{Z})$, there exists a constant C independent on j such that $||h^N f^j|| \leq C 2^{-q(j+1)}$, if $\tilde{h}$ is a stable consistent decimation operator, then the decay rate of the associated prediction error is at least $min(p+1, q)$.

Decay of the prediction error

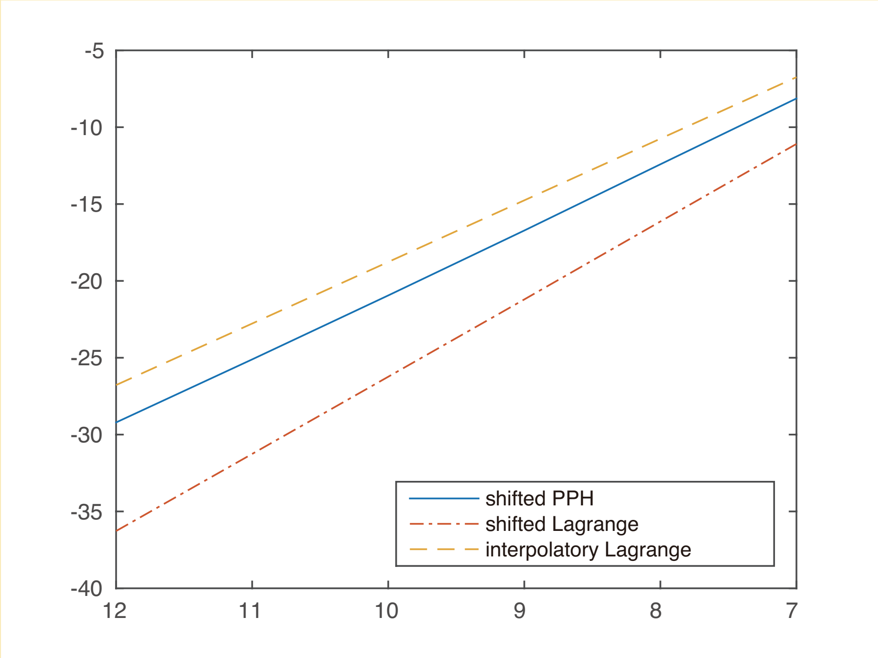


Figure 1: log of the prediction error versus scale from 12 to 7, slope for 4-point interpolatory Lagrange, 4-point shifted Lagrange and 4-point shifted PPH scheme are 4.00717, 5.0379 and 4.21979

4) Properties of multiresolutions; Stability of the decimations

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Proposition. *(Linear decimation) The decimation operator* $\tilde{h}_L$ *is stable if and only if there exists* $i \in \mathbb{N}^*$*, such that the subdivision* h *constructed from sequence* $(2\tilde{h}^i_L)_l, l \in \mathbb{Z}$ *is stable.*

We are then back to a convergence problem for a uniform subdivision.

4) Properties of multiresolutions; Stability of the decimations

Let $h = h_L + h_N$ and $\tilde{h}_L$ be a consistent linear decimation such that $\tilde{h}_L h_N$ is contractive. Then,

Proposition. *(Non linear decimation)*

If there exists a constant $\mu < 1$ such that for all $p \in \mathbb{N}^*$, $h_L^p h_N$ is μ^p Lipschitz then the non linear decimation defined trough the fixed point equation $f^j = \tilde{h}_L f^{j+1} - \tilde{h}_L h_N f^j$ is stable.

4) Numerical applications: performance for image compression

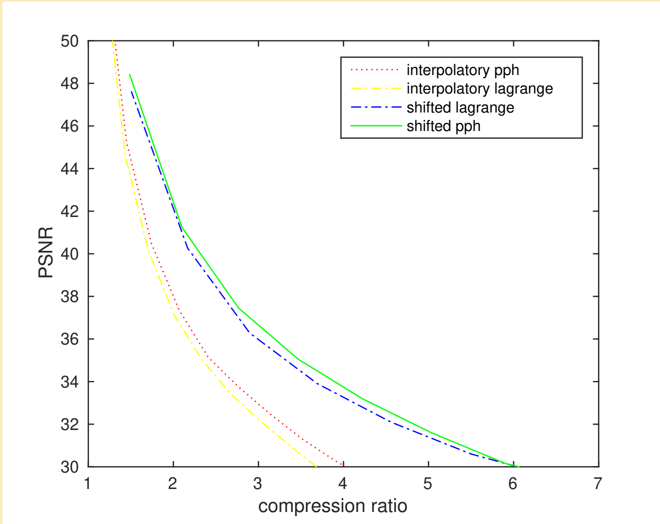


Figure 2: PSNR versus compression ratio for interpolatory Lagrange, shifted Lagrange, interpolatory PPH and shifted PPH multiresolutions.

4) Numerical applications: performance for image compression

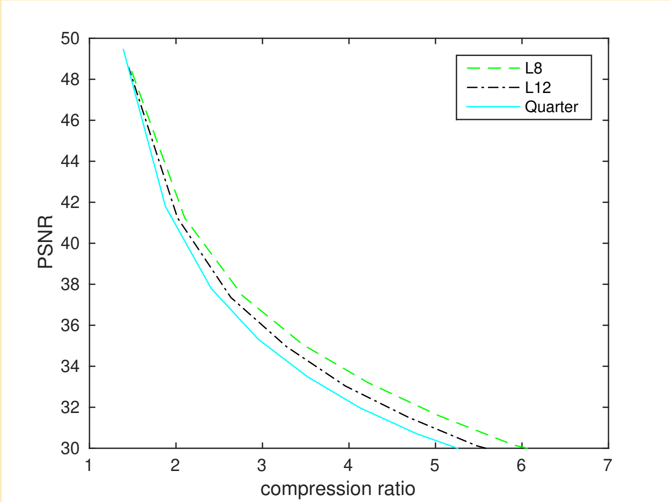


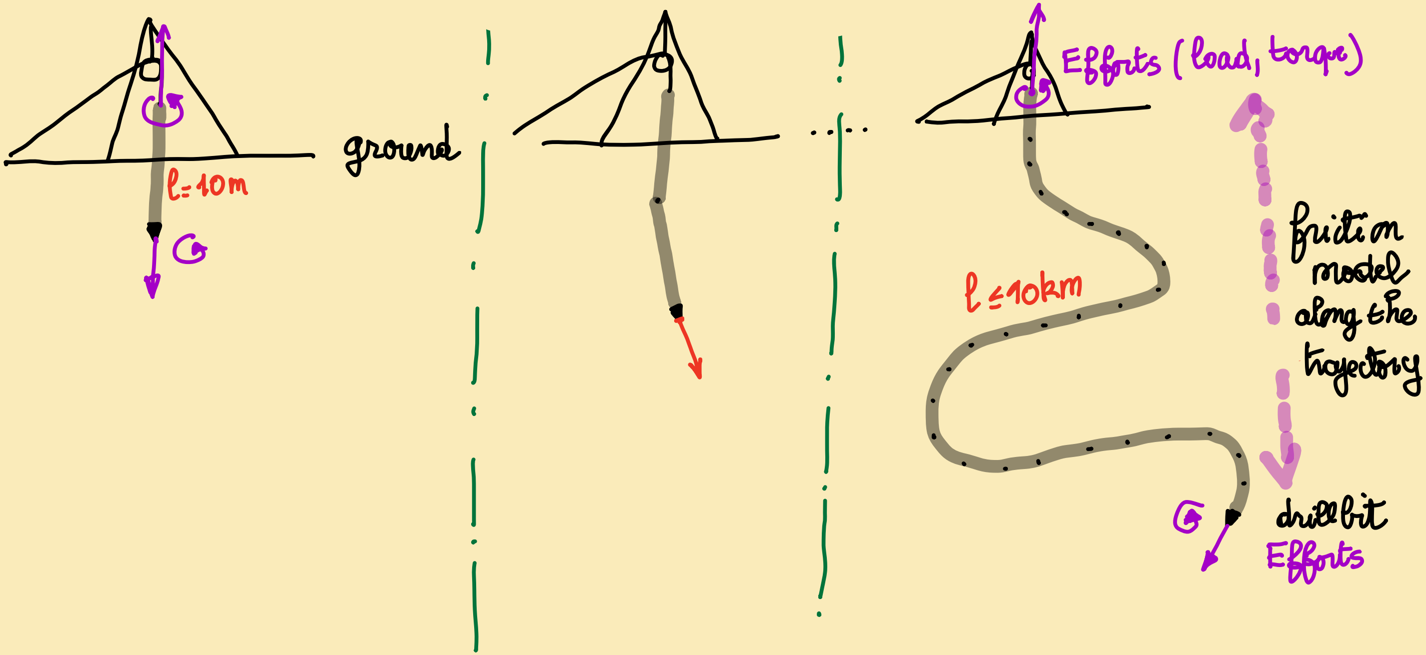
Figure 3: PSNR versus compression ratio for the 4-point shifted PPH subdivision scheme with three different consistent decimation operators

Partial conclusions

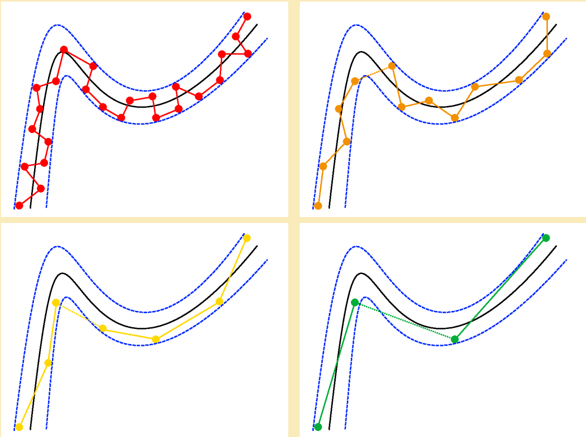
1. General tool for the construction of decimations consistent with linear subdivision,
2. Construction of decimations consistent with non linear subdivision schemes constructed by perturbation,
3. Definition of the details,
4. Properties of the multiresolution,
5. Applications to the Shifted lagrange/PPH schemes.

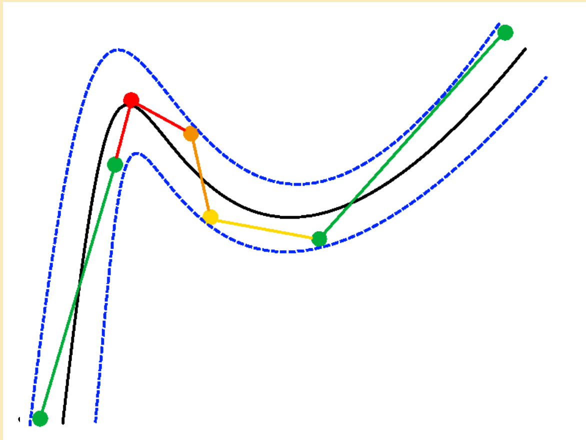
5) Application: Convergence of derivatives

Wellbore monitoring

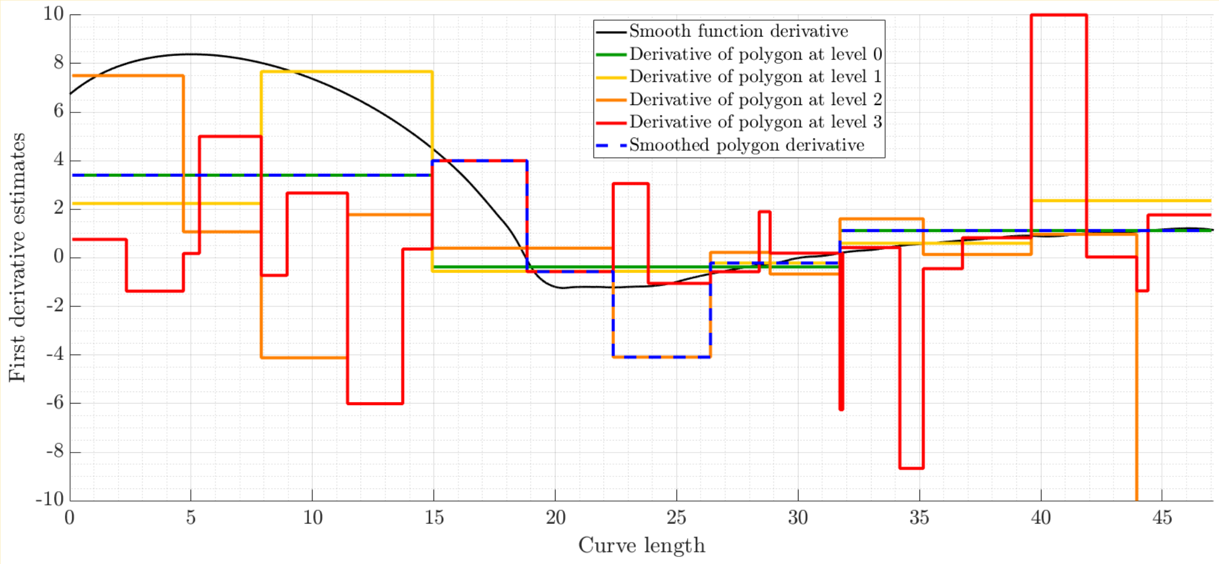


1) Motivation. Convergence of derivatives



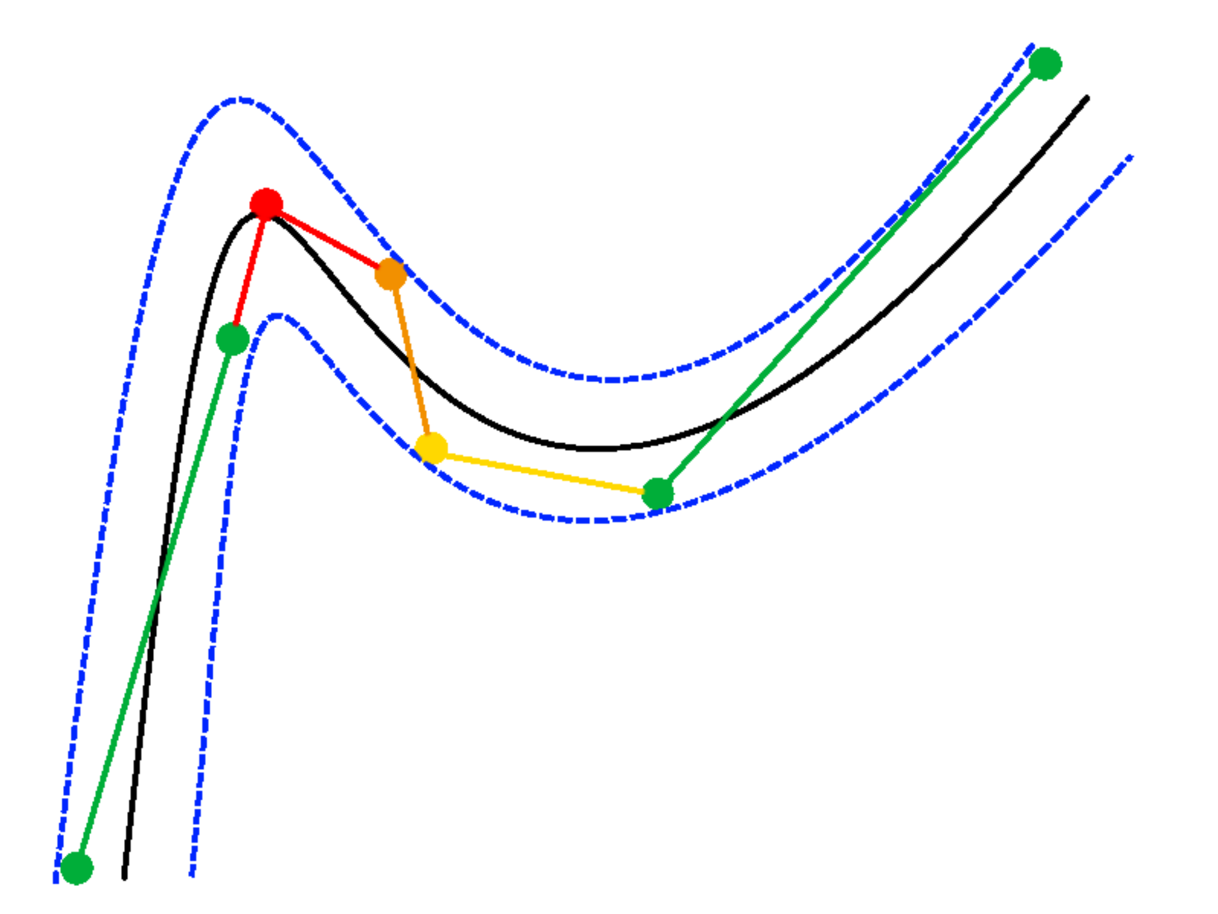


1) Motivation- Convergence of derivatives



1) Convergence of derivatives

- Local scale
- Smoothing



2) Basics properties of subdivisions schemes and multiresolution

- Convergence of the subdivision (h), Ref: Dyn et al. (2004)

$$\forall f^0, \exists f = h^\infty f^0 \in C^0 \text{ such that } \lim_{j \to +\infty} ||f_k^j - f(x_k^j)||_\infty = 0.$$

- Limit function of the subdivision scheme: If $\Phi = h^\infty(\delta^0)$ with $\delta_k^0 = \delta_{k,0}$ then $h^\infty f^0 = \sum_k f_k^0 \Phi(x-k)$. The regularity of Φ defines the regularity of the scheme.

- Stability of the subdivision (h). The operators $\mathbf{h}$ and $\mathbf{h}^{-1}$ are continuous:

  $\mathbf{h} : \{f^0, d^0, d^1, ..., d^j\} \leftrightarrow f^{j+1}$.

3) Basics on divided differences

- $\Delta_1 : X^j \mapsto \Delta_1 X^j$ with$(\Delta_1 X^j)_k = 2^j(X^j_{k+1} - X^j_k)$
- $\Delta_n = \Delta_1^n$
- If $f \in C^\infty(\mathbb{R})$ and $f^j_k = f(k2^{-j})$ then, for some $q \in \{1, 2\}$:

$$(\Delta_n f^j)_k = f^{(n)}(k2^{-j}) + 0(2^{-qj}).$$

4) Key properties

Converging subdivision scheme and finite differences

- **Theorem**(Ref: Dyn (1992)) A subdivision scheme admits a C^m limit function if and only if there exists a converging subdivision scheme for the divided differences Δ_m
- If $X^{j+1} \leftrightarrow \{X^0, e^0, e^1, ...; e^j\}$ then $\Delta_m X^{j+1} \leftrightarrow \{\Delta_m X^0, \Delta_m e^0, \Delta_m e^1, ..., \Delta_m e^j\}$

4) Key properties for a multiresolution of order p

Decay of the details, ϵ-smoothing and local level

- **Theorem**(Ref: Kui et al. (2021)) If $f \in C^\infty$, $f_k^J = f(k2^{-J})$ and if $\mathbf{h}$ is a multiresolution of order p then there exists C_d such that:

$$\forall j \leq J-1, ||d^j|| \leq C_d 2^{-j(p+1)}.$$

- $\tilde{X}_J$ is said to be an ϵ-smoothing of X^J if $\forall j \leq J-1, \tilde{d}_k^j \in \{d_k^j, 0\}$ and $||X_J - \tilde{X}_J||_\infty \leq \epsilon$

- The p-local level of $\tilde{X}^J$ at position $k2^{-J}$is defined as:

$$j_p\left(\tilde{X}^J, k2^{-J}\right) := min\{j \leq J \text{ such that } \left[\forall j' > j, \text{ such that } (j',k') \in C_S(k2^{-J}), \tilde{d}_{k'}^{j'-1} = 0\right]\}.$$

5) $L_{p,\epsilon}$ SMOOTHING DEFINITION: DETAIL TRONCATION

1) *Initialization:* $\tilde{d}^j := d^j$ for all levels $j \in \{J - j_0, \ldots, J-1\}$;

2) For all levels j from highest $(J-1)$ to lowest $(J-j_0)$:

   - For all $\left|\tilde{d}^j_k\right|$ sorted in decreasing order (then starting from highest value):

     (a) Set $\tilde{d}^j_k := 0$;

     (b) *Multiresolution reconstruction:* $\tilde{X}^J$ is constructed from the decomposition given by $\left\{X^{J-j_0}, \tilde{d}^{J-j_0}, \tilde{d}^{J-j_0+1}, \ldots, \tilde{d}^{J-1}\right\}$;

       * If $\left\|\tilde{X}^J - X^J\right\|_\infty < \epsilon$, then proceed with the next $\tilde{d}^{j'}_{k'}$;
       * If not, set back $\tilde{d}^j_k := d^j_k$, then proceed with the next $\tilde{d}^{j'}_{k'}$;

3) *Stopping condition:*

   (a) If step 2) results in no modification of the sequences $\tilde{d}^j$ for all levels $j \in \{J - j_0, \ldots, J-1\}$, then stop;

   (b) Otherwise, repeat steps 2) and 3);

$j_{p,\epsilon}$, the p, ϵ-local level of X^J is defined as the p local level of $L_{p,\epsilon}(X^J)$

5) $L_{p,\epsilon}$ SMOOTHING PROPERTIES

$$f \in C^\infty,\ f^J = \left(f(k2^{-J})\right)_{k\in\mathbb{Z}},\ \left\|f^J - X^J\right\|_\infty < \frac{\epsilon/2}{1 + C_r C_d},$$

- $j_{p,\epsilon}(f^J, k2^{-J}) = -Clog_2(\epsilon)/(p+1)$ decay of the details,
- $||\Delta_n f^J - \Delta_n(L_{p,\epsilon}(f^J)|| \leq C\epsilon^{1-\frac{n}{p+1}}$ details of the multiresolution decomposition of $\Delta_n X^j$,
- **Proposition** There exists a polygon g^J, constructed from a smoothing of X^J, such that $\left\|X^J - g^J\right\|_\infty < \epsilon$, $\left\|f^J - g^J\right\|_\infty < \epsilon$, and whose local levels are at most the local levels of $L_{p,\frac{\epsilon/2}{1+C_r C_d}} f^J$ for all $k \in \mathbb{Z}$. Stability of the multiresolution

5) $L_{p,\epsilon}$ smoothing. Final result

Theorem. *(Ref: Garcia et al. (2021))*

Let $\epsilon > 0$, $K \subseteq I\!R$ *be a compact,* $f \in C^\infty(I\!R)$ *and* $f^J = \left(f(k2^{-J})\right)_{k\in \mathbb{Z}}$ *be the polygon describing* f *at level* J*. Let also* $X^J = \left(X_k^J\right)_{k\in \mathbb{Z}}$ *be a polygon such that* $\left\|f^J - X^J\right\|_\infty < \epsilon$.

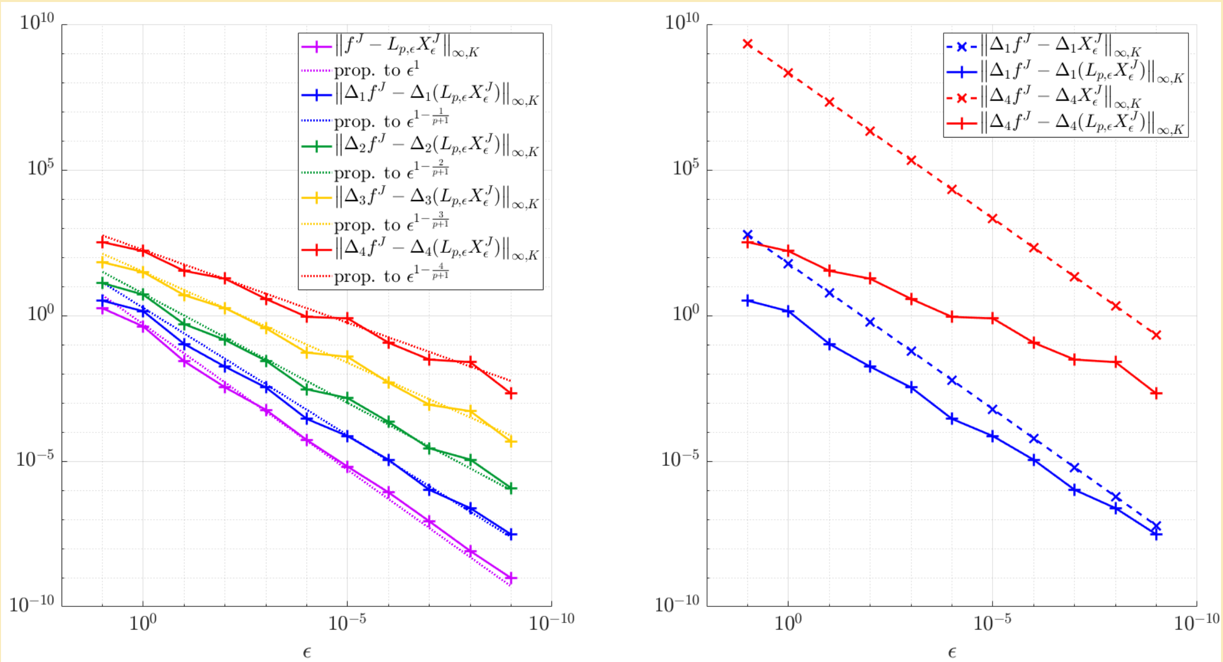
Using a multiresolution analysis of order p *and regularity* $m \le p$*, then, for all integer* n *such that* $n \le m$*:*

$$\left\|f^{(n)} - \Delta_n\left(L_{p,\epsilon}X^J\right)\right\|_{\infty,K} \le C_1 2^{-Jq} + C_2\epsilon^{1-\frac{n}{p+1}} \qquad (0)$$

5) Numerical test

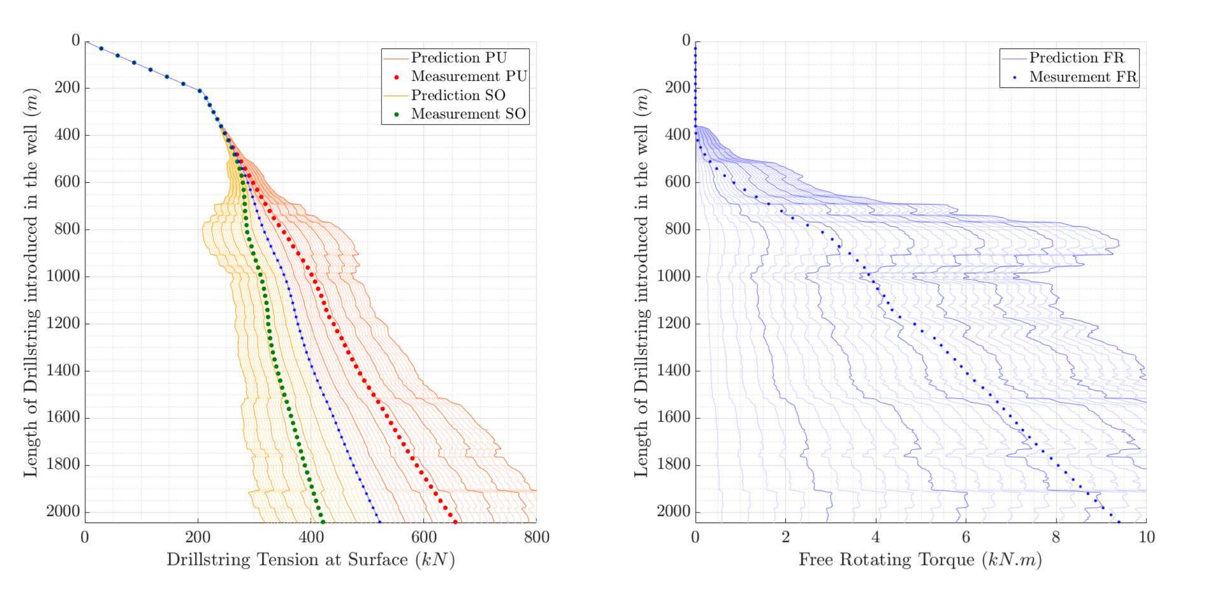
- Subdivision scheme: 8-points shifted Lagrange subdivision scheme ($p = 8$, $m = 4$)
- Associated (non interpolatory) multiresolution
- Expected slope coefficient: $1 - \frac{n}{9}$
- $J \geq 6$
- $CPUtime \simeq 2s$ on a personal computer

5) Numerical test

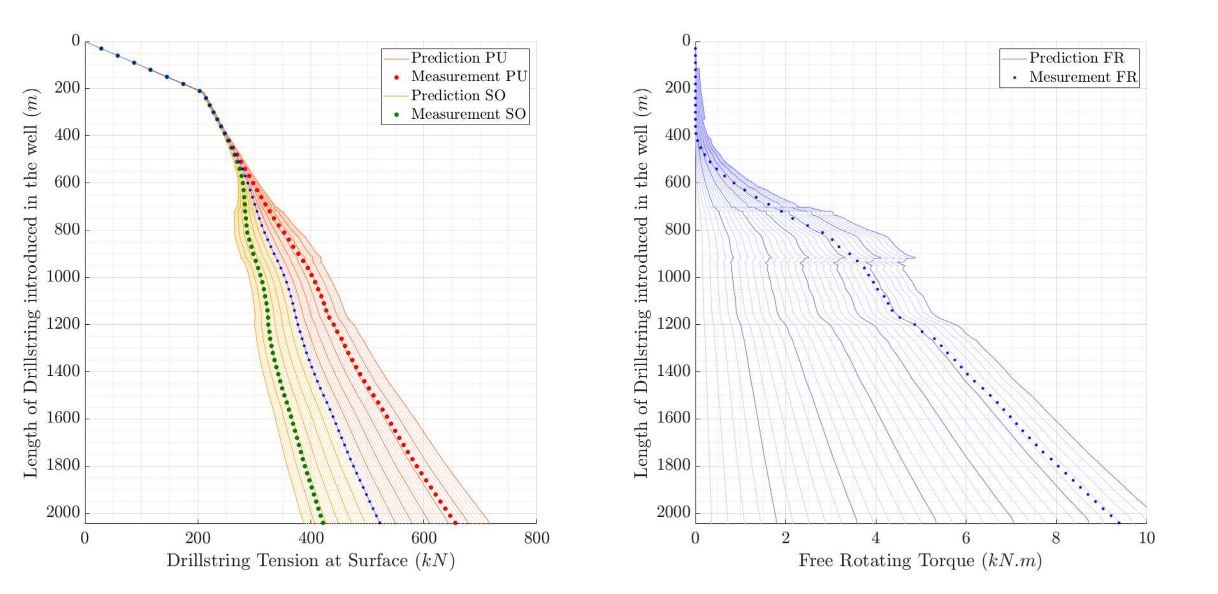


5) Application

Ref: Garcia et al. (2019)



5) Application



5) Conclusions

1. Regular multiresolutions are usefull

   Adaption to finite length interval

   Adaption to non regular sampling

2. Extension to multi dimension (convergence of the normal of a sequence of surfaces)

3. Multiresolution framework for manifold values

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